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ORIGINAL ARTICLE

Reduction the secular solution to periodic solution in the generalized restricted three-body problem

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Abstract The aim of the present work is to find the secular solution around the triangular equilibrium points and reduce it to the periodic solution in the frame work of the generalized restricted thee-body problem. This model is generalized in sense that both the primaries are oblate and radiating as well as the gravitational potential from a belt. We show that the linearized equation of motion of the infinitesimal body around the triangular equilibrium points has a secular solution when the value of mass ratio equals the critical mass value. Moreover, we reduce this solution to periodic solution, as well as some numerical and graphical investigations for the effects of the perturbed forces are introduced. This model can be used to examine the existence of a dust particle near the triangular points of an oblate and radiating binary stars system surrounded by a belt.

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1 Introduction

The problem of three bodies in its most general form means that the three participating bodies are free to move in space and initially move in any given manner under the influence of a given force field. The significance of this problem in space dynamics will appear when the bodies move under the influence of their mutual gravitational attraction according to the Newtonian Law of gravitation. This law specifies that attractive forces between each pair of masses are inversely proportional to the squares of their distances and are proportional to the product masses of the respective particles.

A first consequence of this Law comes when two of the bodies approach each other such that the separation distance between them goes to zero and the force between them also comes to infinity. This circumstance is called double or triple collision according to whether two or three of the participating particles go to the same position in space at the same time. A second consequence of the force law, it follows that when one of the three participating particles is very smaller than the other two. In this situation the motion of the two larger particles will not be influenced by the smaller particle. This dynamical system is referred to as the restricted three-body problem. Therefore, if the motion of the smallest particle is found, we can determine the motion of the other two particles by setting the mass of the smaller particle as zero.

From the above discussion the restricted problem is an abstraction in the physical sense and an approximation in the mathematical sense. Since there is no effect for the smaller mass on the two larger particles unless its mass equals either zero or moves to infinity. Both of these cases reduce the restricted three-body problem to the problem of two bodies.

It is well known that there are five Lagrange solutions in the rotating coordinates system show up as five fixed points at which the infinitesimal body would be stationary if placed there with zero velocity. It is further known that in this rotating coordinates system the infinitesimal body may describe periodic orbits around Lagrange solutions which are referred to as Libration points.

Among the most fundamental questions about motion near libration points are those about the existence of periodic orbits and their stability. Periodic orbits obtain their significance to space mechanics when stable periodic orbits do exist. They may be used as reference orbits. Furthermore, the determination of non-periodic orbits can be performed by perturbation analysis based on periodic orbits. In this section we will survey some results thus for obtained in the investigations of stability and periodicity.

The studies of many authors concerned to the existence of libration points, their stability and the periodic orbits in the framework of the restricted problem under the influence of the lack of spherecity, the photogravitational force, small perturbations in fictitious forces. Some of these works are introduced by Sharma (1987), Elipe and Lara (1997), Ishwar and Elipe (2001), Perdios (2001), Tsirrogiannis et al. (2006), Mittal et al. (2009), Singh and Begha (2011), Abouelmagd et al. (2013), Abouelmagd and El-Shaboury (2012) and Abouelmagd (2012, 2013a, 2013b).

Some researchers devoted their studied for exploring the families of asymmetric periodic orbits. Papadakis (2008) studied the asymmetric solutions of the restricted planar problem of three bodies. He explored numerically the families of asymmetric simple-periodic orbits. He also presented the evolution of these families covering the entire range of the mass parameter of the problem. Furthermore, he regularized the equations of motion of the problem using the Levi-Civita transformations to avoid the singularity due to binary collisions between the third body and one of the primaries. Symmetric relative periodic orbits in the isosceles three-body problem using theoretical and numerical approaches are studied by Shibayama and Yagasaki (2011). They proved that another family of symmetric relative periodic orbits is born from the circular Euler solution besides the elliptic Euler solutions. Their studies also showed that there exist infinitely many families of symmetric relative periodic orbits which are born from heteroclinic connections between triple collisions as well as planar periodic orbits with binary collisions.

Hou and Liu (2011) investigated that the collinear libration points of the real Earth–Moon system are not equilibrium points anymore due to various perturbations. They found special quasi-periodic orbits called dynamical substitutes under the assumption that the Moon's motion is quasi-periodic. In addition they computed the dynamical substitutes of the three collinear libration points in the real Earth–Moon system. In addition, Beevi and Sharma (2012) explored the effect of oblateness of Saturn on the periodic orbits and the regions of quasi-periodic motion around both the primaries in the Saturn-Titan system in the framework of planar circular restricted three-body problem. They studied the effect of oblateness on the location, nature and size of periodic and quasi-periodic orbits, using the numerical technique of Poincare surface of sections. They also showed that some of the periodic orbits change to quasi-periodic orbits due to the effect of oblateness and vice-versa.

Furthermore Abouelmagd and Sharaf (2013) studied and found these orbits around the libration points when the more massive primary is radiating and the smaller is an oblate spheroid. Their study included the effects of zonal harmonic parameters up to 10^{-6} of the main term.

The model of restricted three-body problem when the two primaries are oblate spheroids and radiating as well as the effect of gravitational potential from the belt are constructed by Singh and Taura (2013). They constructed the equations of motion, found the positions of the equilibrium points and examined their linear stability. They also established that, in addition to the usual five equilibrium points, there are two new collinear points L_{n1}, L_{n2} due to the potential from the belt. They investigated that the collinear equilibrium points remain unstable, while the triangular points are stable for $\mu \in (0, \mu_c)$ and unstable for $\mu \in [\mu_c, 1/2]$, where μ_c is the critical mass influenced by the perturbed forces that are aforementioned.

In this work, we will follow Singh and Taura (2013) to find the secular solution around the triangular points in the restricted three-body problem and reduce this solution to periodic solution.

2 Model description

2.1 Hypothesis

We assume that m_1 and m_2 denote the masses of the more massive and the smaller primaries respectively and the mass of the infinitesimal body is m. Let us consider the same assumptions of Singh and Taura (2013). Both masses m_1 and m_2 have circular orbits around their common center of mass. Furthermore m moves in orbital plane under their mutual gravitational fields. The sum of m_1 and m_2 is one where $\mu = m_2/(m_1 + m_2)$ is the mass ratio and the distance between them also is taken as one. In addition the unit of time is chosen to make both the constant of gravitation and the unperturbed mean motion equals unity. Let the origin of the

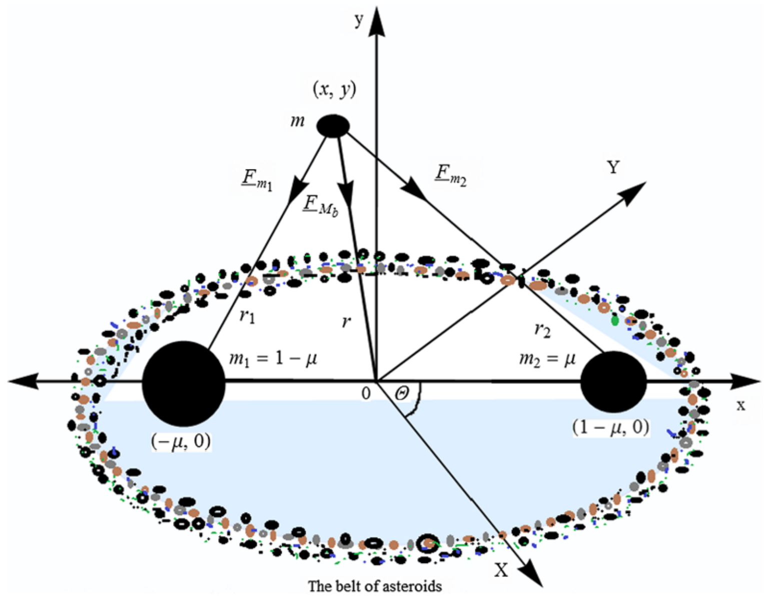

Fig. 1 The configuration of inertial XY and synodic xy coordinate frames of the restricted three-body problem when the primaries are surrounded by the belt of asteroids

sidereal and the synodic coordinates be the common center of mass of the primaries and the synodic coordinates rotate with angular velocity n in positive direction. Hence we can write $m_1 = 1 - \mu$ and $m_2 = \mu \leq 1/2$, the coordinates of m_1, m_2 and m in a synodic frame are $(-\mu, 0, 0)$, $(1-\mu, 0, 0)$ and (x, y, z) respectively, see Fig. 1.

Now let the radiation parameter be $q_i = 1 - p_i$ and the oblateness coefficient is also A_i $(i = 1, 2)$ for the bigger and smaller primaries respectively, where $0 < p_i \ll 1$, $p_i = F_{pi}/F_{gi}$ and $0 < A_i \ll 1$. Moreover, the potential due to the belt is $M_b/(r^2 + T^2)^{1/2}$ see Miyamoto and Nagai (1975) where M_b is the total mass of the belt, r is the radial distance of the infinitesimal body such that $r^2 = x^2 + y^2$, $T = a + b$, a and b are constants that characterize the density profile of the belt. Such that a determine the flatness of the profile and is called the flatness parameter. While b gives the size of the core of the density profile and is known as the core parameter. In the case $a = b = 0$, we obtain the potential of a point mass or spherical subject whose mass is M_b. Furthermore, the directions of forces ($\underline{F}_{m_1}$, $\underline{F}_{m_b}$ and $\underline{F}_{m_1}$) experienced by the mass m can be shown as in Fig. 1.

2.2 The equation of motion

We suppose that m_1 and m_2 move in xy plane. The equations of motion of infinitesimal body given below as in Singh and Taura (2013)

$$\ddot{x} - 2n\dot{y} = \Omega_x \tag{1a}$$

$$\ddot{y} + 2n\dot{x} = \Omega_y \tag{1b}$$

where

$$\Omega = \left\{\frac{1}{2}n^2\left[x^2 + y^2\right] + (1-\mu)q_1\left[\frac{1}{r_1} + \frac{A_1}{2r_1^3}\right]\right.$$
$$\left. + \mu q_2\left[\frac{1}{r_2} + \frac{A_2}{2r_2^3}\right] + \frac{M_b}{(r^2 + T^2)^{1/2}}\right\} \tag{2}$$

where n the perturbed mean motion while r_1 and r_2 are distances of m with respect to m_1 and m_2 respectively, that are given by

$$r_1^2 = (x + \mu)^2 + y^2 \tag{3a}$$

$$r_2^2 = (x + \mu - 1)^2 + y^2 \tag{3b}$$

$$n^2 = 1 + \frac{3}{2}(A_1 + A_2) + \frac{2M_b r_c}{(r_c^2 + T^2)^{3/2}} \tag{4}$$

If we multiply (1a), (1b) by $\dot{x}$ and $\dot{y}$ respectively and add them, we will get a perfect differential for Ω where $\Omega \equiv \Omega(x, y)$ is a function of x, y, hence after integrating we obtain Jacobi integral as

$$\dot{x}^2 + \dot{y}^2 - 2\Omega + c = 0 \tag{5}$$

where c is integration constant.

3 Characteristic equation and its roots

We assume that the infinitesimal body is displaced a little from one of the triangular points (x_0, y_0) to the point $(x_0 + \xi, y_0 + \eta)$ where ξ and η are the variation. Hence the equations of motion and the characteristic equation corresponding to Eqs. (1a) and (1b) will be controlled by

$$\ddot{\xi} - 2n\dot{\eta} = \Omega_{xx}^0\xi + \Omega_{xy}^0\eta \tag{6a}$$

$$\ddot{\eta} + 2n\dot{\xi} = \Omega_{xy}^0\xi + \Omega_{yy}^0\eta \tag{6b}$$